

COARSE AMENABILITY VERSUS PARACOMPACTNESS

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ABSTRACT. Recent research in coarse geometry revealed similarities between certain concepts of analysis, large scale geometry, and topology. Property A of G.Yu is the coarse analog of amenability for groups and its generalization (exact spaces) was later strengthened to be the large scale analog of paracompact spaces using partitions of unity. In this paper we go deeper into divulging analogies between coarse amenability and paracompactness. In particular, we define a new coarse analog of paracompactness modelled on the defining characteristics of expanders.

1. INTRODUCTION

Part of the coarse geometry is driven by ideas from analysis (see [22]) and part of the coarse geometry is driven by dualization of concepts from topology. Thus the class of finite-dimensional spaces was directly dualized by Gromov who created the class of spaces of finite asymptotic dimension (see [13]). On the other hand, Property A of G.Yu [24] came from the analysis side as a coarse version of amenability in order to solve the Novikov Conjecture. Dadarlat and Guentner [9] generalized Property A to the class of exact spaces. Both of these classes coincide when intersected with the class of spaces of bounded geometry. Cencelj, Dydak, and Vavpetič [7] realized that both definitions aim at dualization of paracompactness, so they defined large scale paracompact spaces by strengthening the definition of Dadarlat-Guentner and justified it by creating analogs of theorems characterizing spaces of covering dimension at most n via pushing maps into n -skeleta of simplicial complexes.

This paper is devoted to a more detailed analysis of possible analogs of paracompactness by applying the point of view from [11]: the classical topology investigates topological spaces via their open covers but it is often quite beneficial to apply the tool from analysis, namely partitions of unity, in order to get a more coherent picture of the structure of basic topology. Our approach is to analyze concepts from both points of view (covers versus partitions of unity).

1.1. Approach via covers. First of all, let us start from a useful class in topology, a concept less common outside of general topology; the class of weakly paracompact spaces.

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Definition 1.1. A topological space X is **weakly paracompact** if for each open cover \mathcal{U} of X there is a point-finite open cover \mathcal{V} of X (that means each $x \in X$ belongs to only finitely many elements of \mathcal{V}) that refines \mathcal{U} .

In what follows, the space X is a metric space except where explicitly stated otherwise (e.g. in definitions from general topology). Since r -balls $B(x, r)$ play the role of points at scale r , Definition 1.1 is easily dualizable. First, we need a few concepts.

Definition 1.2. The **diameter** $\text{diam}(\mathcal{U})$ of a family of subsets of X is the infimum of all $\infty \geq r \geq 0$ such that $d_X(x, y) < r$ whenever x, y belong to the same element of \mathcal{U} . If $\text{diam}(\mathcal{U}) < \infty$, we say \mathcal{U} is **uniformly bounded**.

Definition 1.3. The **Lebesgue number** $\text{Leb}(\mathcal{U})$ of a cover of X is the supremum of all $r \geq 0$ such that every r -ball $B(x, r)$ is contained in some element of \mathcal{U} .

A cover at scale r should have Lebesgue number at least r , so here is a dualization of Definition 1.1 (see 4.3 for other, equivalent ways, of dualizing 1.1):

Definition 1.4. X is **large scale weakly paracompact** if for each $r, s > 0$ there is a uniformly bounded cover \mathcal{U} of X of Lebesgue number at least s such that every r -ball $B(x, r)$ is contained in only finitely many elements of \mathcal{U} .

Recall the original definition of paracompactness by Dieudonné [10]:

Definition 1.5. A topological space X is **paracompact** if for each open cover \mathcal{U} of X there is a locally finite open cover \mathcal{V} of X (that means each $x \in X$ has a neighborhood W_x that intersects only finitely many elements of \mathcal{V}) that refines \mathcal{U} .

To dualize 1.5 let us express the meanings of 1.1 and 1.5 in terms of scales: 0-scale is at the level of points and a positive scale is at the level of open covers of X (with refining corresponding to decreasing of the scale, and coarsening corresponding to increasing of the scale). Thus $0 < \mathcal{U}$ means that interiors of elements of \mathcal{U} cover X , and $\mathcal{U} \leq \mathcal{V}$ means that \mathcal{U} is a refinement of \mathcal{V} .

Definition 1.6. Given a cover \mathcal{U} of X and $A \subset X$, by the **horizon** $\text{hor}(A, \mathcal{U})$ of A at scale \mathcal{U} we mean $\{U \in \mathcal{U} \mid A \cap U \neq \emptyset\}$.

Observation 1.7. A topological space X is **weakly paracompact** if for each positive scale \mathcal{U} of X there is positive scale $\mathcal{V} \leq \mathcal{U}$ of X such that each horizon $\text{hor}(x, \mathcal{V})$, $x \in X$, is finite.

Observation 1.8. A topological space X is **paracompact** if for each positive scale \mathcal{U} of X there are positive scales $\mathcal{V} \leq \mathcal{W} \leq \mathcal{U}$ such that the horizon $\text{hor}(V, \mathcal{W})$ of each $V \in \mathcal{V}$ is finite.

Observation 1.9. A topological space X is **compact** if for each positive scale \mathcal{U} of X there is a positive scale $\mathcal{V} \leq \mathcal{U}$ such that the horizon $\text{hor}(X, \mathcal{V})$ of the whole X is finite.

Remark 1.10. Weak paracompactness was first defined by Arens and Dugundji in 1950 [1] as metacompactness and by Bing [4] in 1951 as pointwise paracompactness. Observations 1.7, 1.8, and 1.9 explain the original terminology.

Here is a dualization of 1.5:

Definition 1.11. X has **strong Property A** if for each $s > r > 0$ and each $\epsilon > 0$ there is a uniformly bounded cover \mathcal{U} of X such that for each $x \in X$ the horizon $\text{hor}(B(x, s), \mathcal{U})$ is finite and

$$\frac{|\text{hor}(B(x, r), \mathcal{U})|}{|\text{hor}(B(x, s), \mathcal{U})|} > 1 - \epsilon.$$

In other words, given $x \in X$, the conditional probability of $B(x, r) \cap U \neq \emptyset$ given $B(x, s) \cap U \neq \emptyset$ for some $U \in \mathcal{U}$ can be as close to 1 as we want. We will see later that strong Property A ought to be viewed as a metric analog of non-expanders.

The reason we use in 1.11 the name of strong Property A instead of large scale paracompactness is because the dualization via partitions of unity happened chronologically earlier and we do not know if the two dualizations of paracompactness are identical.

1.2. Approach via partitions of unity. There is another way to define paracompactness (see [12]):

Theorem 1.12. *A topological space X is paracompact if and only if for each open cover \mathcal{U} there is a continuous partition of unity whose carriers refine \mathcal{U} .*

Cencelj-Dydak-Vavpetič [7] realized that the proper dualization of continuity in this case is the concept of a function being (λ, C) -Lipschitz and defined large scale paracompact spaces.

Definition 1.13. A function $f : X \rightarrow Y$ of metric spaces is **(λ, C) -Lipschitz** if

$$d_Y(f(x), f(y)) \leq \lambda \cdot d_X(x, y) + C$$

for all $x, y \in X$.

Definition 1.14. [7] X is **large scale paracompact** if for each $\epsilon > 0$ there is a simplicial partition of unity $f : X \rightarrow l_1(V)$ (see 2.6) satisfying the following conditions:

- a. f is (ϵ, ϵ) -Lipschitz,
- b. the cover of X induced by f (the carriers of f) is uniformly bounded and is a coarsening of the cover of X by $\frac{1}{\epsilon}$ -balls.

The earlier definition of exact spaces by Dadarlat-Guentner is weaker.

Definition 1.15. [9] X is **exact** if for each $r, \epsilon > 0$ there is a partition of unity $f : X \rightarrow l_1(V)$ satisfying the following conditions:

- a. f has (r, ϵ) -variation (that means $d_Y(f(x), f(y)) < \epsilon$ if $d_X(x, y) < r$),
- b. the cover of X induced by f (the carriers of f) is uniformly bounded.

The missing ingredient in 1.15 is the thickness of the cover of X induced by f . The same problem is with the original definition of the Property A of G. Yu. Both work well for the class of spaces of bounded geometry (that means for each r there is an upper bound on the number of points of $B(x, r)$ for all $x \in X$). However, for general metric spaces one needs to make adjustments in order for the theory to work.

In order to unify all the concepts via partitions of unity we created the notion of a **barycentric partition of unity** and we use it to explain and generalize Property A. This is part of our general strategy to explain most concepts via partitions of unity (see [11] for an exposition of basic topology from the point of view of partitions of unity).

2. PARTITIONS OF UNITY

Definition 2.1. $l_1(V)$ is the set of functions $\alpha : V \rightarrow R$ satisfying $\sum_{v \in V} |\alpha(v)| < \infty$.

The subset $\{v \in V | \alpha(v) \neq 0\}$ is called the **carrier** (or **support** of α). Notice it is always countable.

Each $v \in V$ has its **Kronecker delta function** $\delta_v : V \rightarrow R$ which we will quite often identify with v .

Definition 2.2. For each $v \in V$ there is a projection $\pi_v : l_1(V) \rightarrow R$ defined by $\pi_v(\alpha) = \alpha(v)$ (it is the restriction of the evaluation function $R^V \rightarrow R$). By the **open star** $\text{st}(v)$ of $v \in V$ we mean $\pi_v^{-1}(R \setminus \{0\})$. Thus $\{\text{st}(v)\}_{v \in V}$ forms an open cover of non-zero vectors in $l_1(V)$.

Given a non-zero function $f : X \rightarrow l_1(V)$ on a metric space X our general strategy is to measure it both by its Lipschitz number and by the numerical aspects of the cover $\{f^{-1}(\text{st}(v))\}_{v \in V}$ of X (mostly its diameter and its Lebesgue number 1.3).

Definition 2.3. Suppose $f : X \rightarrow l_1(V)$ is a non-zero function on a metric space X and $M > 0$. f is called **M -cobounded** if $\text{diam}(f^{-1}(\text{st}(v))) < M$ for each $v \in V$. f is called **cobounded** if there is $M > 0$ such that f is M -cobounded.

Definition 2.4. Suppose $f : X \rightarrow l_1(V)$ is a non-zero function. The **Lebesgue number** $\text{Leb}(f)$ of f is defined as the Lebesgue number of $\{f^{-1}(\text{st}(v))\}_{v \in V}$ (see 1.3).

Proposition 2.5. Suppose $f : X \rightarrow l_1(V)$ is a non-zero M -cobounded function for some $M > 0$. If $f^{-1}(\text{st}(v)) \neq \emptyset$ for each $v \in V$, then there is an injection $\alpha : V \rightarrow X \times N$ so that the composition $g : X \rightarrow l_1(X \times N)$ of f and the induced linear map $\alpha_* : l_1(V) \rightarrow l_1(X \times N)$ has the property that $g^{-1}(\text{st}(x, n)) \subset B(x, M)$ for all $(x, n) \in X \times N$.

Proof. For each $x \in X$ enumerate all vertices w satisfying $f(x)(w) \neq 0$ as $v(x, 1), v(x, 2), \dots$. For each $w \in V$ pick $x(w) \in f^{-1}(\text{st}(w))$ and then pick the unique $n \in N$ so that $w = v(x(w), n)$. Now set $\alpha(w) = (x(w), n)$.

Since $v(\alpha(w)) = w$, α is injective.

Suppose $y \in g^{-1}(\text{st}(x, n))$. Put $w = v(x, n)$. Therefore $f(y)(w) \neq 0$ and $f(x)(w) \neq 0$ resulting in $x, y \in f^{-1}(\text{st}(w))$ which implies $d(x, y) < M$. Thus $y \in B(x, M)$. \square

Definition 2.6. A **partition of unity** on X is a function $f : X \rightarrow l_1(V)$ such that the l_1 -norm of each $f(x)$, $x \in X$, is 1 and $f(x)(v) \geq 0$ for all $v \in V$.

A partition of unity is called **simplicial** if the carrier of each $f(x) \in l_1(V)$ is finite. f is called **n -dimensional** if the carrier of each $f(x) \in l_1(V)$ contains at most $n+1$ points for each $x \in X$.

The easiest way to create a partition of unity on a set X is to define a non-negative function $f : X \rightarrow l_1(V)$ and then to **normalize** it ($x \mapsto \frac{f(x)}{\|f(x)\|}$).

We need the concept of a **contraction** of a partition of unity.

Definition 2.7. If $f : X \rightarrow l_1(V)$ is a partition of unity and $\alpha : V \rightarrow S$ is a surjection, then by the **contraction** of f along α we mean $\alpha_* \circ f : X \rightarrow l_1(S)$, where $\alpha_* : l_1(V) \rightarrow l_1(S)$ is the induced linear map.

Lemma 2.8. *Suppose g is a contraction of a partition of unity $f : X \rightarrow l_1(V)$.*

a. $\text{Leb}(g) \geq \text{Leb}(f)$.

b. If f is (ϵ, ϵ) -Lipschitz for some $\epsilon > 0$, then g is (ϵ, ϵ) -Lipschitz.

Proof. a. The covering of X induced by g is a coarsening of the cover induced by f . Therefore $\text{Leb}(g) \geq \text{Leb}(f)$.

b. Notice α_* has the norm at most 1 (it is so in view of the Triangle Inequality), hence it is $(1, 0)$ -Lipschitz which implies $\alpha_* \circ f : X \rightarrow l_1(S)$ is (ϵ, ϵ) -Lipschitz. \square

3. BARYCENTRIC PARTITIONS OF UNITY

Definition 3.1. A **barycentric partition of unity** is $f : X \rightarrow l_1(V)$ such that $f(x)$ is of the form $\frac{\chi_{C(x)}}{|C(x)|}$ for each $x \in X$.

Thus f is the normalization of F such that each $F(x)$ is the characteristic function (or the indicator function) of a finite subset $C(x)$ of V .

As each barycentric partition of unity is simplicial, the cover of X induced by them is point-finite.

Definition 3.2. If $\mathcal{U} = \{U_s\}_{s \in S}$ is a point-finite cover of X then its **induced barycentric partition of unity** $p_{\mathcal{U}} : X \rightarrow l_1(S)$ is the normalization of $f(x) = \sum \{\delta_s | x \in U_s\}$.

Thus there is a one-to-one function from point-finite covers of X to barycentric partitions of unity on X . Observe, however, that if $f : X \rightarrow l_1(V)$ is a barycentric partition of unity on X and \mathcal{U} is the cover of X by point-inverses of open stars $\text{st}(v)$, $v \in V$, then $p_{\mathcal{U}}$ may differ from f . Indeed, one may have $f^{-1}(\text{st}(v)) = f^{-1}(\text{st}(w))$ and $v \neq w$.

Lemma 3.3. *For every two non-empty finite subsets A and B of S one has*

$$\frac{|A \Delta B|}{\max(|A|, |B|)} \leq \frac{|A \setminus B|}{|A|} + \frac{|B \setminus A|}{|B|} \leq \left\| \frac{\chi_A}{|A|} - \frac{\chi_B}{|B|} \right\| \leq 2 \cdot \frac{|A \Delta B|}{\min(|A|, |B|)}$$

in $l_1(S)$.

Proof. $A \Delta B := (A \setminus B) \cup (B \setminus A)$ is the **symmetric difference** of A and B . Notice that

$$\| |A| \cdot \chi_B - |B| \cdot \chi_A \| = |A| \cdot |A \setminus B| + |B| \cdot |B \setminus A| + |A \cap B| \cdot ||A| - |B||.$$

Divide both sides by $|A| \cdot |B|$ and perform easy estimations. \square

3.1. Amenability and barycentric partitions of unity. Let us show how amenability of a group can be easily introduced using barycentric partitions of unity.

One can introduce large scale geometry on a group G by declaring uniformly bounded families to be exactly those refining $\{g \cdot F\}_{g \in G}$ for some finite subset $F \subset G$ of G (see Brodskiy-Dydak-Mitra [5]). That structure is metrizable if and only if G is countable and, in case of finitely generated groups, is identical with the coarse structure induced by a word metric on G .

It is natural to consider barycentric partitions of unity on G of the form

$$\phi_F(x) = \frac{\chi_{x \cdot F}}{|F|}.$$

Recall that a **Følner sequence** for a group G is a sequence of finite subsets $F(1) \subset F(2) \subset \dots$ of G such that $\bigcup_{n=1}^{\infty} F(n) = G$ and $\lim_{n \rightarrow \infty} \frac{|gF(n) \Delta F(n)|}{|F(n)|} = 0$ for all $g \in G$.

Proposition 3.4. *Let $F(1) \subset F(2) \subset \dots$ be a sequence of finite subsets of a group G such that for all n the barycentric partition of unity $\phi_{F(n)}$ is (ϵ_n, ϵ_n) -Lipschitz but not (ϵ, ϵ) -Lipschitz for $\epsilon < \epsilon_n$. Then the following conditions are equivalent:*

- a. $\lim_{n \rightarrow \infty} \epsilon_n = 0$,
- b. $\{F(n)\}_{n \geq 1}$ is a Følner sequence.

Proof. Notice $\frac{|xF \Delta yF|}{|F|} = \frac{|x^{-1}yF \Delta F|}{|F|}$ for each $x, y \in G$ and each finite subset F of G . Lemma 3.3 says

$$\frac{|x^{-1}yF(n) \Delta F(n)|}{|F(n)|} = \|\phi_{F(n)}(x) - \phi_{F(n)}(y)\|_1 \leq 2 \cdot \frac{|x^{-1}yF(n) \Delta F(n)|}{|F(n)|}$$

That means a) is equivalent to

$$\lim_{n \rightarrow \infty} \frac{|gF(n) \Delta F(n)|}{|F(n)|} = 0$$

for every $g \in G$. That is the defining condition for a Følner sequence. \square

3.2. Barycentric partitions of unity and Property A. Let us show that Property A of Yu can be defined by replacing arbitrary partitions of unity in 1.15 by barycentric partitions of unity.

Proposition 3.5. *A metric space X has, for every $\epsilon > 0$, an (ϵ, ϵ) -Lipschitz barycentric partition of unity on X that is cobounded, if and only if for each $R > 0$ and each $\epsilon > 0$ there is $S > 0$ and a function A from X to finite subsets of $X \times N$ (N being the set of natural numbers) satisfying the following properties:*

- a. $A(x) \subset B(x, S) \times N$ for each $x \in X$,
- b. if $d(x, y) < R$, then $A(x) \cap A(y) \neq \emptyset$ and

$$\frac{|A(x) \Delta A(y)|}{|A(x) \cap A(y)|} < \epsilon$$

Proof. (\Rightarrow) Let $\epsilon, R > 0$. Let $\bar{\epsilon} = \min\{\epsilon, \frac{1}{2}\} \frac{1}{R+1}$. There exists barycentric $(\bar{\epsilon}, \bar{\epsilon})$ -Lipschitz M -cobounded partition of unity $f: X \rightarrow l_1(V)$. By Proposition 2.5 there exists injection $\alpha: V \rightarrow X \times N$ such that $g^{-1}(\text{st}(x, n)) \subset B(x, M)$ for all $(x, n) \in X \times N$, where $g = \alpha_* \circ f$. Let $S = \alpha(V)$. Then $g: X \rightarrow l_1(S)$ is contraction of f along α and it is also M -bounded barycentric partition of unity. By Lemma 2.8 g is $(\bar{\epsilon}, \bar{\epsilon})$ -Lipschitz. Let $A(x) = \{(y, n) \in X \times N \mid g(x)(y, n) \neq 0\}$. Because g is barycentric, $A(x)$ is finite for all x . Because $g^{-1}(\text{st}(x, n)) \subset B(x, M)$ for all $(x, n) \in X \times N$, $A(x) \subset B(x, M) \times N$ for all $x \in X$.

Let $d(x, y) < R$. If $|A(x) \cap A(y)| < \frac{1}{2}|A(x)|$, then

$$\frac{1}{2} < \frac{|A(x) - A(y)|}{|A(x)|} \leq \|g(x) - g(y)\| \leq \bar{\epsilon}d(x, y) + \bar{\epsilon} < \bar{\epsilon}(R+1) < \frac{1}{2}$$

which is a contradiction. Therefore $|A(x)| < 2|A(x) \cap A(y)|$ for $d(x, y) < R$ in particular $A(x) \cap A(y) \neq \emptyset$. By Lemma 3.3

$$\begin{aligned} \frac{|A(x) \Delta A(y)|}{|A(x) \cap A(y)|} &\leq \frac{|A(x) \Delta A(y)|}{2 \max\{|A(x)|, |A(y)|\}} \leq \left\| \frac{\chi_{A(x)}}{|A(x)|} - \frac{\chi_{A(y)}}{|A(y)|} \right\| = \\ \|g(x) - g(y)\| &\leq \bar{\epsilon} d(x, y) + \bar{\epsilon} < \bar{\epsilon}(R + 1) \leq \epsilon. \end{aligned}$$

(\Leftarrow) Let $\epsilon > 0$. By assumption there is a function A from X to finite subsets of $X \times N$ such that $A(x) \subset B(x, S) \times N$ for each $x \in X$ for some $S > 0$ and for $d(x, y) < \frac{2-\epsilon}{\epsilon}$ the intersection $A(x) \cap A(y) \neq \emptyset$ and $\frac{|A(x) \Delta A(y)|}{|A(x) \cap A(y)|} < \frac{\epsilon}{2}$. Then $f : X \rightarrow l_1(X \times N)$ defined as $f(x) = \frac{\chi_{A(x)}}{|A(x)|}$ is S -cobounded barycentric partition of unity. If $d(x, y) \geq \frac{2-\epsilon}{\epsilon}$ then

$$\|f(x) - f(y)\| \leq 2 = \epsilon \frac{2-\epsilon}{\epsilon} + \epsilon \leq \epsilon d(x, y) + \epsilon.$$

If $d(x, y) < \frac{2-\epsilon}{\epsilon}$ then by Lemma 3.3

$$\|f(x) - f(y)\| \leq 2 \frac{|A(x) \Delta A(y)|}{\min\{|A(x)|, |A(y)|\}} \leq 2 \frac{|A(x) \Delta A(y)|}{|A(x) \cap A(y)|} < 2 \frac{\epsilon}{2} \leq \epsilon d(x, y) + \epsilon,$$

therefore f is (ϵ, ϵ) -Lipschitz. \square

3.3. Creation of barycentric partitions of unity.

Definition 3.6. If $f : X \rightarrow l_1(V)$ is a partition of unity, then by an **expansion** of f we mean any partition of unity g so that f is its contraction.

Proposition 3.7. Suppose $f : X \rightarrow l_1(V)$ is a cobounded partition of unity that is (ϵ, ϵ) -Lipschitz for some $\epsilon > 0$. If f is the normalization of an integer-valued function $F : X \rightarrow l_1(V)$ (that means $F(x)(v) \in \mathbb{Z}_+$ for all $(x, v) \in X \times V$), then there is a barycentric expansion g of f that is cobounded, (ϵ, ϵ) -Lipschitz, and $\text{Leb}(g) = \text{Leb}(f)$.

Proof. Let $S = \{(v, n) \in V \times N \mid F(x)(v) \leq n \text{ for some } x \in X\}$, and let $\alpha : S \rightarrow V$ be the projection onto the first coordinate. Define $G : X \rightarrow l_1(S)$ by $G(x)(v, i) = 1$ if $F(x)(v) \geq i$ and $G(x)(v, i) = 0$ if $F(x)(v) \leq i$. Then $\sum_{i \in N} G(x)(v, i) = F(x)(v)$ for all $x \in X$ and $v \in V$, therefore $\|G(x)\| = \sum_{(v, i) \in S} G(x)(v, i) = \sum_{v \in V} F(x)(v) = \|F(x)\|$. Let g be the normalization of G . Then f is the contraction of g along α and

$$\begin{aligned} \|f(x) - f(y)\| &= \sum_{v \in V} \left| \frac{F(x)(v)}{\|F(x)\|} - \frac{F(y)(v)}{\|F(y)\|} \right| = \\ &= \sum_{v \in V} \left| \frac{\sum_j G(x)(v, j)}{\|G(x)\|} - \frac{\sum_j G(y)(v, j)}{\|G(y)\|} \right| = \\ &= \sum_{v \in V} \sum_j \left| \frac{G(x)(v, j)}{\|G(x)\|} - \frac{G(y)(v, j)}{\|G(y)\|} \right| = \\ &= \|g(x) - g(y)\| \end{aligned}$$

for all $x, y \in X$. Hence g is (ϵ, ϵ) -Lipschitz. Because $f^{-1}(\text{st}(v)) = g^{-1}(\text{st}(v, 1)) \supset g^{-1}(\text{st}(v, n))$ for every $n \in N$, g is cobounded and $\text{Leb}(g) = \text{Leb}(f)$. \square

Proposition 3.8. *If X is separable at scale r (that means there is a countable subset S of X with $B(S, r) = X$), then there is a $3r$ -cobounded barycentric partition of unity f on X whose Lebesgue number is at least r .*

Proof. Enumerate elements of S as x_1, x_2, \dots . Put $U_n = B(x_n, 2r)$, $V_n = U_n \setminus \bigcup_{i=1}^{n-1} U_i$, and $W_n = B(V_n, r)$. Notice $\{V_n\}$ is a cover of X , so the Lebesgue number of $\mathcal{W} = \{W_n\}$ is at least r .

Given $x \in X$ choose $m \geq 1$ so that $d(x, x_m) < r$. Notice $B(x, r) \subset B(x_m, 2r)$, so $B(x, r) \cap V_n = \emptyset$ for all $n > m$. Hence $x \notin W_n$ for all $n > m$.

Let $f = p_{\mathcal{W}}$ and it is clear f is a $3r$ -cobounded barycentric partition of unity on X whose Lebesgue number is at least r . \square

4. LARGE SCALE PARACOMPACTNESS

Exercise 4.1. A topological space X is weakly paracompact if and only if for each open cover \mathcal{U} of X there is a barycentric partition of unity f on X so that the cover of X induced by f is open and refines \mathcal{U} .

Proposition 4.2. *If X coarsely embeds in a large scale weakly paracompact space Y , then X is large scale weakly paracompact.*

Proof. Suppose $f : X \rightarrow Y$ is a coarse embedding. Given $r, s > 0$ find $r', s' > 0$ with the following properties:

- a. $d_X(x, y) < r$ implies $d_Y(f(x), f(y)) < r'$,
- b. $d_Y(f(x), f(y)) < s'$ implies $d_X(x, y) < s$.

Pick a uniformly bounded cover \mathcal{U} of Y of Lebesgue number at least s' such that every r' -ball $B(z, r')$ is contained in only finitely many elements of \mathcal{U} . Define \mathcal{V} as $f^{-1}(\mathcal{U})$ and observe \mathcal{V} is of Lebesgue number at least s such that every r -ball $B(x, r)$ is contained in only finitely many elements of \mathcal{V} . \square

Proposition 4.3. *The following conditions are equivalent for each metric space X :*

- a. *For each $r > 0$ there is a uniformly bounded cover \mathcal{U} of X such that every r -ball $B(x, r)$ intersects only finitely many elements of \mathcal{U} .*
- b. *X is large scale weakly paracompact.*
- c. *For every uniformly bounded cover \mathcal{U} of X there exists uniformly bounded point-finite cover \mathcal{V} such that \mathcal{U} is refinement of \mathcal{V} .*
- d. *For each $M > 0$ there exists a cobounded barycentric partition of unity $f : X \rightarrow l_1(V)$ of Lebesgue number at least M .*
- e. *For each $M > 0$ there exists a cobounded simplicial partition of unity $f : X \rightarrow l_1(V)$ of Lebesgue number at least M .*

Proof. a) \implies b). Suppose $r, s > 0$. Pick a cover \mathcal{V} such that every $(r + s)$ -ball $B(x, r + s)$ intersects only finitely many elements of \mathcal{V} . Notice $B(x, r) \subset B(x, r + s)$ implies $B(x, r + s) \cap A \neq \emptyset$ for any subset A of X . Therefore, the family $\mathcal{U} := \{B(V, s) \mid V \in \mathcal{V}\}$ is a uniformly bounded cover of X of Lebesgue number at least s such that every r -ball $B(x, r)$ is contained in only finitely many elements of \mathcal{U} . According to Definition 1.4, X is large scale weakly paracompact.

b) \implies c). Suppose \mathcal{U} is a uniformly bounded cover of X . Put $r = \text{diam}(\mathcal{U}) + 1$, $s = 2r$, and find a uniformly bounded cover \mathcal{W} of X of Lebesgue number at least s such that every r -ball $B(x, r)$ is contained in only finitely many elements of \mathcal{W} .

Given $A \subset X$ define $B(X, -r)$ as $X \setminus B(X \setminus A, r)$ and observe $x \in B(A, -r) \implies B(x, r) \subset A$. Therefore, the family $\mathcal{V} := \{B(W, -r) | W \in \mathcal{W}\}$ is a uniformly bounded cover of X of Lebesgue number at least r such that every $x \in X$ is contained in only finitely many elements of \mathcal{V} . Also, \mathcal{V} coarsens \mathcal{U} .

c) \implies d). Given $M > 0$ there is a point-finite uniformly bounded cover \mathcal{V} such that $\{B(x, M) | x \in X\}$ is a refinement of \mathcal{V} . The standard partition of unity $p_{\mathcal{V}}$ is cobounded simplicial and of Lebesgue number at least M .

d) \implies e) is obvious.

e) \implies a). Let $r > 0$. There exists a cobounded simplicial partition of unity $f : X \rightarrow l_1(S)$ of Lebesgue number at least $r + 1$. Consider $\mathcal{U} = \{B(\text{st}(s), -r) | s \in S\}$. It is a uniformly bounded cover of X such that every r -ball $B(x, r)$ intersects only finitely many elements of \mathcal{U} as $B(x, r) \cap B(A, -r) \neq \emptyset \implies x \in A$. \square

Corollary 4.4. *If X is large scale separable, then it is large scale weakly paracompact.*

Proof. X is large scale separable if there is a countable set S of X such that $X = B(S, r)$ for some $r > 0$. Let $M > 0$ and $\bar{M} = \max\{M, r\}$. Then $B(S, \bar{M}) = X$ and by Proposition 3.8 there exists a cobounded partition of unity on X whose Lebesgue number is at least $\bar{M} \geq M$. By Proposition 4.3 X is large scale weakly paracompact. \square

Problem 4.5. *Is every metric space large scale weakly paracompact?*

Use 4.3 to prove the following.

Corollary 4.6. *Every large scale paracompact space X is large scale weakly paracompact.*

We do not know if we can weaken Definition 1.14 by dropping the assumption of partitions of unity being simplicial.

Problem 4.7. *Let X be a metric space such that for each $\epsilon > 0$ there is a partition of unity $f : X \rightarrow l_1(V)$ satisfying the following conditions:*

- a. *f is (ϵ, ϵ) -Lipschitz,*
- b. *the cover of X induced by f (the carriers of f) is uniformly bounded and is a coarsening of the cover of X by $\frac{1}{\epsilon}$ -balls.*

Is X large scale paracompact?

We will show the answer to 4.7 is positive if X is large scale weakly paracompact.

Lemma 4.8. *Suppose $1 > \epsilon > 0$. If $f : X \rightarrow l_1(V)$ is an $(\frac{\epsilon}{2}, \frac{\epsilon}{2})$ -Lipschitz partition of unity on X that is cobounded, then there is a simplicial partition of unity $g : X \rightarrow l_1(V)$ that is (ϵ, ϵ) -Lipschitz and is cobounded.*

Proof. For each $x \in X$ pick a finite subset $C(x)$ of the carrier of $f(x)$ such that

$$\sum_{v \notin C(x)} f(x)(v) < \frac{\epsilon}{4}.$$

Define $g(x)$ by setting $g(x)(v) = 0$ for all $v \notin C(x)$, then picking $v(x) \in C(x)$ and setting $g(x)(v(x)) = f(x)(v(x)) + \sum_{v \notin C(x)} f(x)(v)$. For $v \in C(x) \setminus \{v(x)\}$ we put $g(x)(v) = f(x)(v)$. \square

We are ready to show that the difference between exact spaces of Dadarlat-Guentner [9] (see 1.15) and large scale paracompact spaces is large scale weak paracompactness.

Theorem 4.9. *If X is large scale weakly paracompact and for each $\epsilon > 0$ there is an (ϵ, ϵ) -Lipschitz partition of unity on X that is cobounded, then X is large scale paracompact.*

Proof. Given $\epsilon > 0$ pick a cover $\{U_s\}_{s \in S}$ of X consisting of non-empty sets that is M -cobounded and every ball $B(x, \frac{1}{\epsilon})$ intersects only finitely many elements of $\{U_s\}_{s \in S}$. For each $s \in S$ pick $x_s \in U_s$.

For each $x \in X$ let $S(x) = \{s \in S \mid B(x, \frac{1}{\epsilon}) \cap U_s \neq \emptyset\}$.

Pick $\delta < \frac{\epsilon}{2M+1}$ and pick a simplicial partition of unity $f : X \rightarrow l_1(V)$ on X that is cobounded and (δ, δ) -Lipschitz using 4.8.

Define a new partition of unity g on X by the formula

$$g(x) = \frac{\sum_{s \in S(x)} f(x_s)}{|S(x)|}.$$

Notice it is cobounded.

Given $x \in X$ choose $s \in S$ so that $x \in U_s$ and choose $v \in V$ satisfying $f(x_s)(v) \neq 0$. If $y \in B(x, \frac{1}{\epsilon})$, then $s \in S(y)$ so $g(y)(v) \neq 0$ and $y \in g^{-1}(\text{st}(v))$. That proves the Lebesgue number of g is at least $\frac{1}{\epsilon}$.

Given $x, y \in X$,

$$|S(x)| \cdot |S(y)| \cdot (g(x) - g(y)) = \sum_{s \in S(x)} |S(y)| \cdot f(x_s) - \sum_{t \in S(y)} |S(x)| \cdot f(x_t)$$

can be rewritten as the sum of $|S(x)| \cdot |S(y)|$ differences of the form

$$f(x_s) - f(x_t)$$

where $s \in S(x)$ and $t \in S(y)$. Therefore $d(x_s, x_t) < 2M + d(x, y)$ implying $\|f(x_s) - f(x_t)\| \leq \delta(2M + d(x, y)) + \delta$. Thus

$$|S(x)| \cdot |S(y)| \cdot |g(x) - g(y)| \leq |S(x)| \cdot |S(y)| \cdot (\delta(2M + d(x, y)) + \delta)$$

resulting in

$$\|g(x) - g(y)\| \leq \delta(2M + d(x, y)) + \delta < \epsilon \cdot d(x, y) + \epsilon$$

as we can assume $M > 1/2$. □

Corollary 4.10. *If X coarsely embeds in a large scale paracompact space Y , then X is large scale paracompact.*

Proof. By 4.2 X is large scale weakly paracompact. Suppose $f : X \rightarrow Y$ is a coarse embedding. Pick a sequence $f_n : X \rightarrow l_1(V_n)$ of cobounded partitions of unity that are $(\frac{1}{n}, \frac{1}{n})$ -Lipschitz and observe that $g_n = f_n \circ f$ is a sequence of cobounded partitions of unity such that for some sequence $\epsilon_n \rightarrow 0$, g_n is (ϵ_n, ϵ_n) -Lipschitz. □

5. PROPERTY A

Problem 5.1. *Is X large scale paracompact if it has, for every $\epsilon > 0$, an (ϵ, ϵ) -Lipschitz barycentric partition of unity on X that is cobounded?*

Remark 5.2. In view of 4.9 it suffices to show X is large scale weakly paracompact in order to answer 5.1 in the positive.

We want to generalize Yu's [24] definition of Property A to arbitrary metric spaces so that spaces with Property A are large scale paracompact.

Definition 5.3. A metric space X has **Property A** if for every $\epsilon > 0$ there is an (ϵ, ϵ) -Lipschitz barycentric partition of unity on X that is cobounded and whose Lebesgue number is at least $\frac{1}{\epsilon}$.

Observation 5.4. *As in 4.10 one can show that if X coarsely embeds in a space Y with Property A, then X has Property A.*

Definition 5.5. A metric space X is **large scale finitistic** if for every $r > 0$ there is a uniformly bounded cover \mathcal{U} of X whose Lebesgue number $\text{Leb}(\mathcal{U})$ is at least r and there is $n(\mathcal{U}) = n > 0$ such that each $x \in X$ belongs to at most n elements of \mathcal{U} .

Here is a generalization of doubling spaces from analysis ([15], p.81).

Definition 5.6. A metric space X is **coarsely doubling** (or **large scale doubling**) if there is $M > 0$ such that for every $r > M$ there is a natural number $n(r)$ such that every $2r$ -ball can be covered by at most $n(r)$ set of r -balls.

Proposition 5.7. *a. Every space of bounded geometry is doubling.
b. Every coarsely doubling space X is large scale finitistic.
c. Every coarsely doubling space X contains a subspace Y of bounded geometry such that the inclusion $Y \rightarrow X$ is a coarse equivalence.*

Proof. a. Every space of bounded geometry is doubling (which means the condition in 5.6 holds for all scales r).

b) and c). Suppose there is $M > 0$ such that for every $r > M$ there is a natural number $n(r)$ such that every $2r$ -ball can be covered by at most $n(r)$ -element set of r -balls.

Assume $r > 2M$. Choose a maximal subset $Y = \{x_s\}_{s \in S}$ of X with the property that $d(x_s, x_t) \geq r$ for each $s \neq t$ in S . Given $x \in X$ consider $T = \{s \in S \mid x_s \in B(x, 2r)\}$. Notice $|T| \leq n(\frac{r}{2}) \cdot n(r)$ as otherwise $B(x, 2r)$ cannot be covered by a set of at most $n(\frac{r}{2}) \cdot n(r)$ -element $\frac{r}{2}$ -balls (that would result in two elements x_s, x_t , $s, t \in T$, to end up in the same element of the cover). That means the horizon of x in $\{B(x_s, 2r)\}_{s \in S}$ contains at most $n(\frac{r}{2}) \cdot n(r)$ elements and X is large scale finitistic due to $\text{Leb}(\{B(x_s, 2r)\}_{s \in S}) \geq r$.

Use Y as above for $r = 2M + 1$. Put $r(m) = 2^{m-1} \cdot r$ for $m \geq 1$. Notice that $B(x, r(m+1)) \cap Y$ contains at most $n(\frac{r}{2}) \cdot n(r) \cdot \dots \cdot n(r(m))$ points for all $m \geq 1$. For $m = 1$ it has been just proved. For general m it follows by induction. \square

The following theorem generalizes known results on Property A for spaces of bounded geometry (see [23], [14]) and spaces of finite asymptotic dimension (see [7] and [8]).

Theorem 5.8. *A large scale finitistic metric space X has Property A if and only if it is large scale paracompact.*

Proof. Suppose X is large scale finitistic and large scale paracompact. Given $\epsilon > 0$ we will find an (ϵ, ϵ) -Lipschitz simplicial partition of unity $f : X \rightarrow l_1(V)$ that is cobounded and $\text{Leb}(f) \geq \frac{1}{\epsilon}$.

Pick a uniformly bounded cover $\{U_s\}_{s \in S}$ of multiplicity at most $n + 1$ which is a coarsening of the cover of X induced by f . Let $\alpha : V \rightarrow S$ be defined so that

$$f^{-1}(\text{st}(v)) \subset U_{\alpha(v)}$$

for each $v \in V$. We may assume α is surjective by removing elements U_t of the cover $\{U_s\}_{s \in S}$ such that $t \in S \setminus \alpha(V)$.

Consider the contraction $g = \alpha_* \circ f : X \rightarrow l_1(S)$ of f . Notice it is (ϵ, ϵ) -Lipschitz by 2.8, cobounded, $\text{Leb}(g) \geq \frac{1}{\epsilon}$, and g is n -dimensional (in view of $g^{-1}(\text{st}(s)) \subset U_s$ for each $s \in S$).

Find a natural number $m \geq \frac{2(n+1)}{\epsilon} + (n+1) \cdot (n+2)$. Consider $G = m \cdot g$ and express it as the sum $G_1 + G_2$, where $G_2 \geq 0$ is integer-valued, $G_2(x)(v) > 0$ iff $G(x)(v) > 0$, $\|G_2(x)\| = m$ for each $x \in X$, and $\|G_1(x)\| \leq 2n + 2$ for each $x \in X$. The way to do it is to set initially $G_1(x)(v)$ to be equal to $G(x)(v) - 1$ if $0 < G(x)(v) < 1$, $G_1(x)(v) = 0$ if $G(x)(v) = 0$, and $G_1(x)(v) = G(x)(v) - \lfloor G(x)(v) \rfloor$ if $G(x)(v) \geq 1$ (here $\lfloor q \rfloor$ is the integer part of x).

Let $k(x) = \sum_{v \in V} G_1(x)(v)$ and $G_2 = G - G_1$. Notice $k(x)$ is an integer-valued function and $|k(x)| < n+1$. For every $x \in X$ proceed as follows. If $k(x) < 0$, then there is $w \in V$ with $G_2(x)(w) > |k(x)|$, in which case we assign $G_1(x)(w) - k(x)$ as the new $G_1(x)(w)$ and we assign $G_2(x)(w) + k(x)$ as the new $G_2(x)(w)$. If $k(x) \geq 0$, then we pick any $w \in V$ such that $G(x)(w) > 0$ and we assign $G_1(x)(w) - k(x)$ as the new $G_1(x)(w)$ and we assign $G_2(x)(w) + k(x)$ as the new $G_2(x)(w)$.

Let $h : X \rightarrow l_1(S)$ be the normalization of G_2 . Notice $|h(x) - g(x)| \leq \frac{2n+2}{m} < \epsilon$ for each $x \in X$, so h is $(2\epsilon, 2\epsilon)$ -Lipschitz. Also, it induces the same cover of X as g does which implies h is cobounded and $\text{Leb}(h) \geq \frac{1}{\epsilon}$. h can be expanded to a barycentric partition of unity p by 3.7 that is $(2\epsilon, 2\epsilon)$ -Lipschitz, is cobounded and $\text{Leb}(p) \geq \frac{1}{\epsilon}$. \square

6. STRONG PROPERTY A

Observation 6.1. *As in 4.2 one can show that if X coarsely embeds in a space Y with strong Property A, then X has strong Property A.*

Proposition 6.2. *The following conditions are equivalent for every metric space X ;*

- a. X has strong Property A.
- b. For each $r > 0$ and each $\epsilon > 0$ there is a uniformly bounded cover \mathcal{U} of X such that for each $x \in X$ the horizon $\text{hor}(B(x, 2r), \mathcal{U})$ is finite and

$$\frac{|\text{hor}(B(x, r), \mathcal{U})|}{|\text{hor}(B(x, 2r), \mathcal{U})|} > 1 - \epsilon.$$

- c. For each $r > 0$ and each $\epsilon > 0$ there is a uniformly bounded cover \mathcal{U} of X such that for each $x \in X$ the horizon $\text{hor}(B(x, r), \mathcal{U})$ is finite and

$$\frac{|\text{hor}(x, \mathcal{U})|}{|\text{hor}(B(x, r), \mathcal{U})|} > 1 - \epsilon.$$

d. For each $s > r > 0$ and each $M, \epsilon > 0$ there is a uniformly bounded cover \mathcal{U} of X of Lebesgue number at least M such that for each $x \in X$ the horizon $\text{hor}(B(x, s), \mathcal{U})$ is finite and

$$\frac{|\text{hor}(B(x, r), \mathcal{U})|}{|\text{hor}(B(x, s), \mathcal{U})|} > 1 - \epsilon.$$

Proof. a) \implies b) and d) \implies a) are obvious.

b) \implies c). Given $r > 0$ and $\epsilon > 0$ pick a uniformly bounded cover $\mathcal{V} = \{V_t\}_{t \in T}$ of X such that for each $x \in X$ the horizon $\text{hor}(B(x, 2r), \mathcal{V})$ is finite and

$$\frac{|\text{hor}(B(x, r), \mathcal{V})|}{|\text{hor}(B(x, 2r), \mathcal{V})|} > 1 - \epsilon.$$

Define $U_t = B(V_t, r)$ and put $\mathcal{U} = \{U_t\}_{t \in T}$. Notice $B(x, r) \cap B(V_t, r) \neq \emptyset$ implies $B(x, 2r) \cap V_t \neq \emptyset$. That means

$$\text{hor}(B(x, r), \mathcal{U}) \subset \text{hor}(B(x, 2r), \mathcal{V}).$$

Similarly, $\text{hor}(x, \mathcal{U}) = \text{hor}(B(x, r), \mathcal{V})$. Therefore

$$\frac{|\text{hor}(x, \mathcal{U})|}{|\text{hor}(B(x, r), \mathcal{U})|} > 1 - \epsilon.$$

c) \implies d). Suppose $s > r > 0$ and $M, \epsilon > 0$. Pick a uniformly bounded cover $\mathcal{V} = \{V_t\}_{t \in T}$ of X such that for each $x \in X$ the horizon $\text{hor}(B(x, s + M), \mathcal{V})$ is finite and

$$\frac{|\text{hor}(x, \mathcal{V})|}{|\text{hor}(B(x, s + M), \mathcal{V})|} > 1 - \epsilon.$$

Define $U_t = B(V_t, M)$ and put $\mathcal{U} = \{U_t\}_{t \in T}$. Notice $B(x, s) \cap B(V_t, M) \neq \emptyset$ implies $B(x, s + M) \cap V_t \neq \emptyset$. That means

$$\text{hor}(B(x, s), \mathcal{U}) \subset \text{hor}(B(x, s + M), \mathcal{V}).$$

Since $\text{hor}(x, \mathcal{V}) \subset \text{hor}(B(x, r), \mathcal{U})$,

$$\frac{|\text{hor}(B(x, r), \mathcal{U})|}{|\text{hor}(B(x, s), \mathcal{U})|} > 1 - \epsilon.$$

□

Proposition 6.3. *Every space X with strong Property A has Property A.*

Proof. Given $\epsilon > 0$ consider $r, \mu > 0$ to be determined later and pick $\delta > 0$ so that

$$1 + \mu > \frac{1}{1 - \delta}.$$

Then pick a uniformly bounded cover \mathcal{U} of X such that $\text{Leb}(\mathcal{U}) \geq 4r$ (see 6.2) and for each $x \in X$ the horizon $\text{hor}(B(x, 2r), \mathcal{U})$ is finite and

$$\frac{|\text{hor}(B(x, r), \mathcal{U})|}{|\text{hor}(B(x, 2r), \mathcal{U})|} > 1 - \delta.$$

For each $x \in X$ let $A(x) = \text{hor}(B(x, 2r), \mathcal{U})$ and $D(x) = \text{hor}(B(x, r), \mathcal{U})$.

Define the barycentric partition of unity $g : X \rightarrow l_1(S)$ as the normalization of the function $f(x) = \chi_{A(x)}$. Notice that $\text{Leb}(g) \geq 2r$.

If $d(x, y) < r$, then $D(x) \subset A(x) \cap A(y)$. Thus

$$|A(x)| < (1 + \mu) \cdot |D(x)| \leq (1 + \mu) \cdot |A(x) \cap A(y)|$$

resulting in

$$|A(x) \setminus A(y)| < \mu \cdot |A(x) \cap A(y)|.$$

Using 3.3 we get that $d(x, y) < r$ implies $\|g(x) - g(y)\| < 4\mu$.

If we request $r > \frac{1}{\epsilon}$, we get $\text{Leb}(g) > \frac{1}{\epsilon}$. If we request $\mu < \frac{\epsilon}{4}$ and $r > \frac{2-\epsilon}{\epsilon}$, then we get g is (ϵ, ϵ) -Lipschitz. Indeed, in case $d(x, y) \geq r$ it is automatic $(\epsilon \cdot d(x, y) + \epsilon) > 2 \geq \|g(x) - g(y)\|$ in this case), and $d(x, y) < r$ implies $\|g(x) - g(y)\| < 4\mu \leq \epsilon$. \square

Theorem 6.4. *For a coarsely doubling metric space X the following conditions are equivalent:*

- a. X is large scale paracompact,
- b. X has Property A,
- c. X has strong Property A.

Proof. a) \equiv b) follows from 5.8. In view of 6.3 it suffices to show b) \implies c).

Using 5.7 we can reduce it to X of bounded geometry. Suppose $s > 0$. Pick $M > 0$ so that each s -ball $B(x, s)$, $x \in X$, contains at most M points.

Given any $\mu > 0$ find a uniformly bounded cover $\mathcal{U}(\mu)$ such that the barycentric partition of unity $p_{\mathcal{U}(\mu)}$ induced by $\mathcal{U}(\mu)$ is (μ, μ) -Lipschitz.

Given $x \in X$ let $A(x) = \text{hor}(x, \mathcal{U}(\mu))$. By 3.3 it implies

$$|A(x) \Delta A(y)| < (s+1) \cdot \mu \cdot \max(|A(x)|, |A(y)|)$$

whenever $d(x, y) < s$. Therefore

$$|A(y)| < \frac{|A(x)|}{1 - (s+1) \cdot \mu}$$

whenever $d(x, y) < s$.

Enumerate all points $y \in B(x, s)$ as y_1, \dots, y_k for some $k \leq M$. Now

$$\begin{aligned} \left| \bigcup_{i=1}^k A(y_i) \right| &\leq |A(x) \cup \bigcup_{i=1}^k (A(y_i) \setminus A(x))| \leq |A(x)| + \sum_{i=1}^k |A(x) \Delta A(y_i)| \leq \\ &\leq |A(x)| + \frac{M \cdot (s+1) \cdot \mu \cdot |A(x)|}{1 - (s+1) \cdot \mu} = \left(1 + \frac{M \cdot (s+1) \cdot \mu}{1 - (s+1) \cdot \mu}\right) \cdot |A(x)| \end{aligned}$$

Given any $\epsilon > 0$ we may choose $\mu > 0$ so that $(1 + \frac{M \cdot (s+1) \cdot \mu}{1 - (s+1) \cdot \mu})^{-1} > 1 - \epsilon$. Notice

$$\bigcup_{i=1}^k A(y_i) = \text{hor}(B(x, s), \mathcal{U}(\mu)). \text{ Since } A(x) = \text{hor}(x, \mathcal{U}(\mu)),$$

$$\frac{|\text{hor}(x, \mathcal{U}(\mu))|}{|\text{hor}(B(x, s), \mathcal{U}(\mu))|} > 1 - \epsilon.$$

Thus X has strong Property A by 6.2. \square

Corollary 6.5. *The Hilbert space is not large scale paracompact.*

Proof. As shown in [2], the Hilbert space contains a bounded geometry subspace (the box space of the free group of two generators) that does not have Property A. Hence the Hilbert space cannot be large scale paracompact. \square

Remark 6.6. One cannot derive 6.5 from earlier result of P.Nowak [20] (who constructed a subspace of the Hilbert cube without Property A) as he used a weaker definition of Property A than we do and his subspace is not of bounded geometry.

Problem 6.7. *Find a direct/simple proof of the Hilbert space not being large scale paracompact.*

7. EXPANDERS AND STRONG PROPERTY A

Let G be an undirected graph with vertex set $V(G)$ and edge set $E(G)$. For a collection of vertices $A \subseteq V(G)$, let ∂A denote the collection of all edges going from a vertex in A to a vertex outside of A :

$$\partial A := \{(x, y) \in E \mid x \in A, y \in V(G) \setminus A\}.$$

(Remember that edges are unordered, so the edge (x, y) is the same as the edge (y, x) .)

Definition 7.1. The **Cheeger constant** of a finite graph G , denoted $h(G)$, is defined by

$$h(G) := \min \left\{ \frac{|\partial A|}{|A|} \mid A \subseteq V(G), 0 < |A| \leq \frac{|V(G)|}{2} \right\}.$$

The Cheeger constant is strictly positive if and only if G is a connected graph. Intuitively, if the Cheeger constant is small but positive, then there exists a "bottle-neck", in the sense that there are two "large" sets of vertices with "few" links (edges) between them. The Cheeger constant is "large" if any possible division of the vertex set into two subsets has "many" links between those two subsets (see Wikipedia).

Definition 7.2. A finite graph G is a (k, ε) -**expander** if each vertex of G has valency at most k , and $h(G) \geq \varepsilon > 0$.

A sequence of finite graphs $\{G_i\}$ is called an **expander sequence** if $|G_i| \rightarrow \infty$ and there exists k, ε such that each G_i is a (k, ε) -expander.

Expander sequences were defined by Bassalygo and Pinsker in 1973 [3]. It is not obvious that such sequences exist. Their existence was first proved by Pinsker [21], in a non-constructive way. Margulis was the first to give explicit examples of expanders using discrete groups with property (T) [18], [19]. For more on expanders see [16].

Proposition 7.3. *For any expander sequence $\{G_i\}$ there is $c > 0$ with the property that for any subset A of $V(G_i)$ with $|A| \leq \frac{|V(G_i)|}{2}$ the number of points not in A such that their 2-ball intersects A is at least $c \cdot |A|$.*

Proof. Suppose each G_i is a (k, ε) -expander. The collection ∂A of all edges going from a vertex in A to a vertex outside of A has at least $\varepsilon \cdot |A|$ elements. Their endpoints not in A form exactly the set C of points not in A such that their 2-ball intersects A . Since each point $c \in C$ can produce at most k edges in ∂A , $|C| \geq \frac{|\partial A|}{k} \geq \varepsilon \frac{|A|}{k}$, and $c = \frac{\varepsilon}{k}$ works. \square

Proposition 7.4. *Any expander sequence does not have strong Property A.*

Proof. Pick a uniformly bounded cover $\mathcal{U} = \{U_s\}_{s \in S}$ of an expander sequence such that for each x

$$\frac{|\text{hor}(B(x, 1), \mathcal{U})|}{|\text{hor}(B(x, 2), \mathcal{U})|} > p.$$

Restrict the cover to the graph $G = G_m$, with m sufficiently large for the number of elements in each U_s to be less than half of the vertices of G_m .

Let P be the set of pairs (x, s) such that $x \notin U_s$ but $B(x, 2)$ intersects U_s . Let c

be a positive constant as in 7.3. By fixing s and counting points $x \in U_s$ such that $(x, s) \in P$, we see that

$$|P| \geq c \cdot \sum_{s \in S} |U_s|.$$

Also,

$$|P| \leq \frac{1-p}{p} \cdot \sum_{s \in S} |U_s|.$$

Indeed,

$$\begin{aligned} |P| &= \sum_{x \in G} (|\text{hor}(B(x, 2), \mathcal{U})| - |\text{hor}(B(x, 1), \mathcal{U})|) < \\ &< \frac{1-p}{p} \sum_{x \in G} |\text{hor}(B(x, 1), \mathcal{U})| = \frac{1-p}{p} \cdot \sum_{s \in S} |U_s|. \end{aligned}$$

Therefore

$$c \leq \frac{1-p}{p}$$

and there is a bound on p from above

$$p \leq \frac{1}{1+c}.$$

□

Remark 7.5. See [17] for a proof of the fact expander sequences do not have Property A using cohomology. See [6] for a cohomology characterization of Property A.

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